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Quantum Electrodynamics at Large Distances III: Verification of Pole Factorization and the Correspondence Principle *

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Abstract

In two companion papers it was shown how to separate out from a scattering function in quantum electrodynamics a distinguished part that meets the correspondence-principle and pole-factorization requirements. The integrals that define the terms of the remainder are here shown to have singularities on the pertinent Landau singularity surface that are weaker than those of the distinguished part. These remainder terms therefore vanish, relative to the distinguished term, in the appropriate macroscopic limits. This shows, in each order of the perturbative expansion, that quantum electrodynamics does indeed satisfy the pole-factorization and correspondence-principle requirements in the case treated here. It also demonstrates the efficacy of the computational techniques developed here to calculate the consequences of the principles of quantum electrodynamics in the macroscopic and mesoscopic regimes.

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Disclaimer

1. Introduction

In papers I[1] and II[2] we examined the functions $F(g)$ associated with the infinite set of graphs $\{g\}$ obtained by dressing a simple triangle graph G with soft photons in all possible ways. A distinguished set of contribution to these functions $F(g)$ was singled out and called "dominant" because these contributions were expected to dominate the macroscopic behaviour of the scattering functions. Each of these distinguished parts was shown to be well defined, and to have a singularity of the form $\log\varphi$ on the (Landau-Nakanishi) triangle-diagram singularity surface $\varphi = 0$. This form $\log\varphi$ agrees with the form of the singularity of the original Feynman function $F(G)$ on $\varphi = 0$, and it produces the same kind of large-distance fall-off. Moreover, these dominant contributions yield (exactly once) every term in the perturbative expansion of the triangle-diagram version of the pole-factorization property

$$\mathrm{Disc}F'|_{\varphi=0} = F'_1 F'_2 F'_3.$$

The left-hand side of this equation represents value at $\varphi = 0$ of the discontinuity across the surface $\varphi = 0$ of the function F' that is obtained by omitting the contributions from the "classical" photons. These latter contributions are supplied by the unitary operator $U(L)$ — after the transformation to coordinate space. Consequently, the discontinuity formula given above entails that the contributions from these "dominant" terms give just the classical-type large-distance behaviour demanded by the correspondence principle: the rate of fall-off at large distances is exactly what follows from the classical concept of three stable charged particles, each moving from one scattering region to another, and the electromagnetic field generated by $U(L)$ is exactly the quantum analog of the classical electromagnetic field generated by the motions of these three charged particles. In the present article we shall show that, in each order of the perturbation expansion, the terms of the remainder give *no* contributions to the discontinuity defined above. Consequently, these "non-dominant" terms give no contribution to the leading term in the the asymptotic large-distance behaviour, and hence the correspondence-principle requirement is satisfied.

In section 2 we examine the simplest example, namely the triangle graph G dressed with *one* internal soft photon. The remainder part is separated into a sum of terms. For some terms the weakening of the singularity on the surface

$\varphi = 0$ is associated with the topological complexity of the graph that represents this term, namely its non-separability: cutting the graph at the three $*$ lines associated with the three Feynman-denominator poles does not separate the graph into three disjoint parts. This means that the integration over the momenta of the internal photons tends to shift the position of the singularity, and hence weaken it. For the remaining terms the weakening of the singularity on $\varphi = 0$ is due to the replacement of one or more of the three pole singularities $(p_s^2 - m^2)^{-1}$ of the integrand by a pair of logarithmic singularities: this replacement of the pole singularities in the integrand by logarithmic singularities likewise leads to a weakening of the singularity of the integral on $\varphi = 0$.

Our problems here are first to show that these reductions in the degree of the singularity on $\varphi = 0$, which emerge easily within our formalism in this simple one-photon example, hold for every g obtained by dressing the triangle graph G with soft photons, and second to show that the weakening of the singularity is, in every case, a weakening by at least one full power of φ, up to a prescribed finite number of powers of $\log \varphi$ that increases linearly with the number of photons, and hence with powers of the coupling constant. This strong result means that the validity of the correspondence–principle in the large–scale limit, which is established here at each order of the perturbative expansion, cannot be upset by an accumulation of powers of $(\log \varphi)^n$ that leads to a singularity of the form $\varphi^{-\beta}$, where β is of the order of the fine-structure constant ($\sim 1/137$). Accumulations of this kind occur often in field theories. The appearance of the fine-structure constant in the exponent arises from the fact that usually, just as in our case, the number of powers of $\log \varphi$ is linearly tied to the number of powers of the fine-structure constant. We have not studied, in our case, the numerical factors that multiply the terms of the remainder, except to show that they are all finite. Hence we can make here no claim pertaining to meaningfulness of the infinite sum in our case: we plan to examine this question later.

To establish our general conclusions we need two auxiliarly results. The first is a geometric property concerning the structure of the Landau-Nakanishi surface. It is proved in section 3. The second pertains to several singular integrals. The needed computation is performed in section 4. The required properties of the various intergrals are then proved in sections 5, 6, and 7.

2. Examination of non-dominant singularities for the one-photon case.

Let us consider the contributions associated with the graph g shown in Figure 1.

In references 1 and 2 we showed how to separate the contribution represented by the graph of Fig. 1 into a meromorphic part consisting of a sum of the four terms represented by the the four $*$ graphs of Figure 2 , plus a "non-meromorphic" part.

The term associated with the graph a of Figure 2 is separable, and is classified as dominant. The associated function F_a is given by (4.1) and (4.3) of Ref. 1, and has a logarithmic singularity along the Landau surface $\varphi = 0$. We also claimed there that the other three terms in the meromorphic part are are infrared finite, and have weaker singularities along the Landau surface $\varphi = 0$. In the following subsection we verify this claim for the case of the function F_b associated with graph (b) of Fig. 2. The other two cases, c and d, can be treated similarly.

Figure 1: A graph g representing a soft-photon correction to a hard-photon triangle-diagram process G. The letters Q near the ends of the wiggly line that represents the soft photon indicate that this particle is coupled to the charged particle through the “quantum” part of the full quantum-electrodynamical coupling.

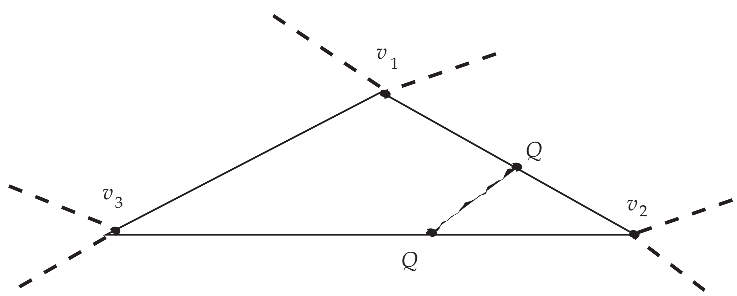
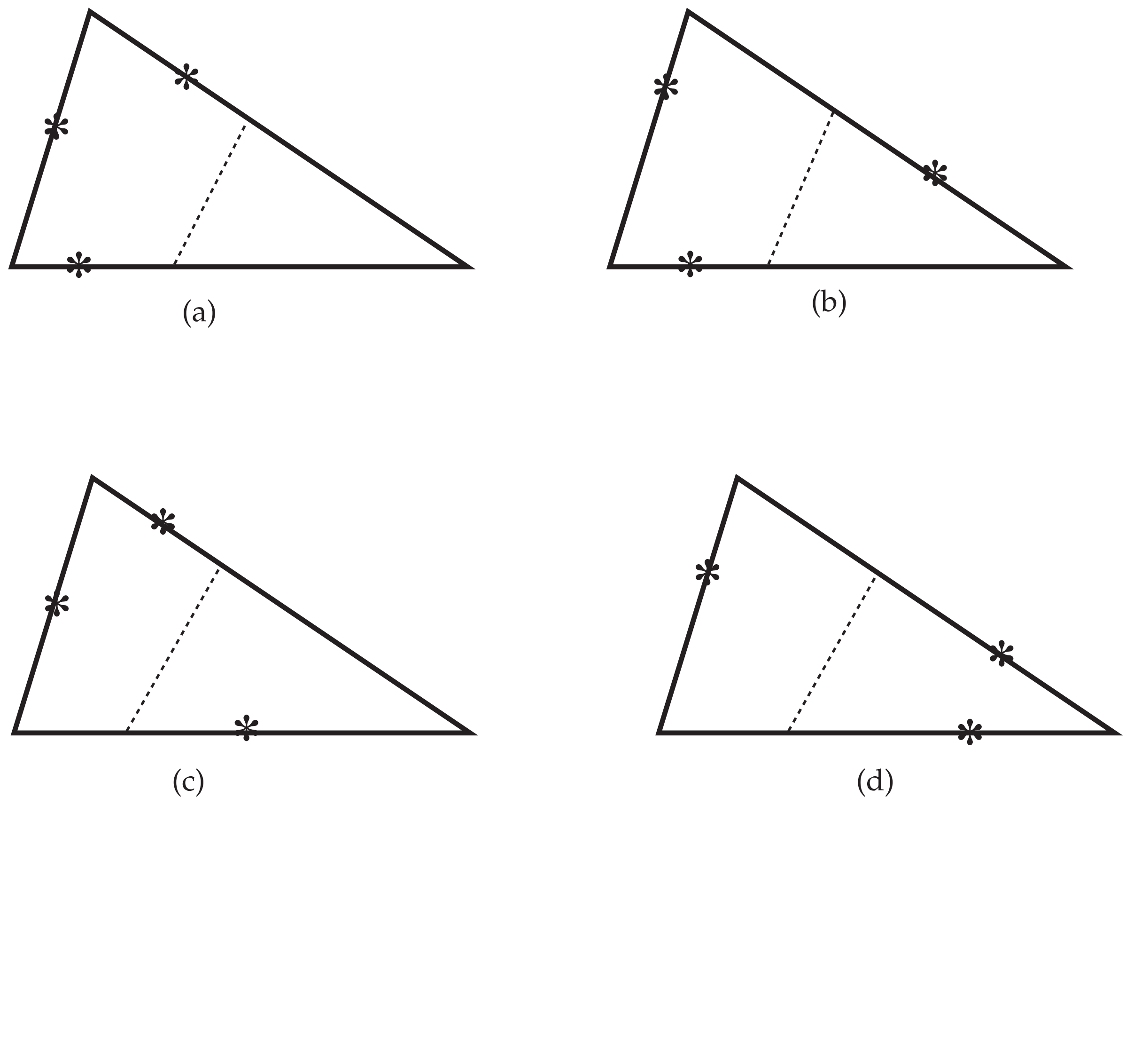


Figure 2: Four $*$ graphs representing the four terms in the meromorphic part of the function represented by the graph in fig. 1. These four terms arise from a decomposition of the meromorphic parts associated with each of the three sides of the triangle into poles times residues. The $*$ lines represent Feynman-denominator poles. The other charged-particle lines represent residue factors.



2.i. The contribution from a one-photon nonseparable meromorphic part.

The function F_b was given in (4.4) of Ref. 1:

$$
\begin{aligned}
F_b \; = & \int \frac{d^4 p}{(2\pi)^4} \int_0^\delta 2r dr \int \frac{d^4\Omega}{(2\pi)^4} \quad \frac{i\delta(\Omega_0^2 + \vec{\Omega}^2 - 1)}{\Omega^2 + i0} \\
& \mathrm{Tr} \left\{ \frac{i(\not{p} + m)}{p^2 - m^2} V_1 \Big(\frac{(2p_{i\mu} + 2r\Omega_\mu)\Omega^2 (2p_1\Omega + 2r\Omega^2)^{-1} - \not{\Omega}\gamma_\mu}{2p_1\Omega + r\Omega^2} \Big) \right. \\
& \times \frac{(\not{p}_1 + r\not{\Omega} + m)}{(p_1 + r\Omega)^2 - m^2} V_2 \\
& \times \left. \Big(\frac{2p_{2\mu}\Omega^2(2p_2\Omega) - \gamma_\mu \not{\Omega}}{2p_2\Omega + r\Omega^2} \Big) \frac{(\not{p}_2 + m)}{p_2^2 - m} V_3 \right\} \qquad (2.ii.1)
\end{aligned}
$$

It was shown in Ref. 2 that we can distort the Ω-contour so that $Im\Omega^2 > 0$ at $\Omega^2 = 0$, and $Imp_j\Omega > 0$ $(j = 1, 2)$ at $p_j\Omega = 0$. Then, except for three pole-factors $p^2 - m^2, (p_1 + r\Omega)^2 - m^2$ and $p_2^2 - m^2$, each denominator of the integrand of F_b is different from zero.

The r-integration $\int_0^\delta rdr/[(p_1 + r\Omega)^2 - m^2]$ can be explicitly performed, and when $p_1\Omega \neq 0$ its dominant singularity along $p_1^2 = m^2$ is

$$
-\frac{(p_1^2 - m^2)\log(p_1^2 - m^2)}{4(p_1\Omega)^2}.
$$

Combining this singularity, instead of the ordinary pole $1/(p_1^2 - m^2)$, with the other two poles, i.e., $1/(p^2 - m^2)$ and $1/(p_2^2 - m^2)$, we perform the p-integration and find a singularity $A(q, \Omega)\varphi(q)^2 \log\varphi(q)$, with A being analytic. Performing the Ω-integration along the compact distorted contour, the dominant singularity of F_b is $\varphi^2 \log\varphi$.

Essentially the same argument covers the case where one of the two meromorphic parts is due to a C-coupling. Then the factor rdr becomes simply dr, and the singularity becomes $\varphi \log\varphi$.

2.ii. The contribution from a pair of non-meromorphic parts arising from one photon.

The contribution of I in (4.6) in Ref. 1 to the amplitude is

$$F = \int_{|\Omega|=1} \frac{d^4\Omega}{\Omega^2 + i0} \int_0^\delta \frac{dr}{r} \int d^4 p_3 \frac{1}{p_3^2 - m^2 + i0} \log \frac{(q_1 + p_3 + r\Omega)^2 - m^2 + i0}{(q_1 + p_3)^2 - m^2 + i0}$$
$$\times \log \frac{(p_3 - q_3 - r\Omega)^2 - m^2 + i0}{(p_3 - q_3)^2 - m^2 + i0}, \qquad (2.ii.1)$$

with the p_i defined as in Fig. 1 of ref.2. Here the Ω-contour is deformed so that $Im\Omega^2 > 0$ and $Imp_j\Omega > 0$ (j=1,2). Performing the p_3-integration we find

$$\int_{|\Omega|=1} \frac{d^4\Omega}{\Omega^2 + i0} \int_0^\delta \frac{dr}{r} (G(q_1 + r\Omega, q_3 + r\Omega)$$
$$- G(q_1 + r\Omega, q_3) - G(q_1, q_3 + r\Omega) + G(q_1, q_2)). \quad (2.ii.2)$$

where

$$G(q_1, q_2) = \varphi(q_1, q_2)^2 \log(\varphi(q_1, q_2) + i0).$$

Since

$$(\partial/\partial r)\varphi(q_1 + r\Omega, q_3 + r\Omega) = (\partial\varphi/\partial q_1 + \partial\varphi/\partial q_3) \cdot \Omega$$
$$= (\alpha_1 \rho_1 + \alpha_2 \rho_2) \cdot \Omega = -\alpha_3 p_s \cdot \Omega \neq 0$$

holds by the Landau equation, we can find non-vanishing functions $a(q_1, q_3, r, \Omega)$ and $b(q_1, q_3)$ for which

$$\varphi(q_1 + r\Omega, q_3 + r\Omega) = a(q_1, q_3, r, \Omega)(r - b(q_1, q_3)\varphi(q_1 + r\Omega, q_3))$$

holds (Cf.§ 3 below). Similar decompositions hold also for $\varphi(q_1 + r\Omega, q_2)$ and $\varphi(q_1, q_2 + r\Omega)$. Hence application of the results in § 4 below to $\int_0^\delta \frac{dr}{r}(G(q_1 + r\Omega, q_3 + r\Omega) - G(q_1, q_3))$ etc. entails that the r-integration in (2.ii.2) produces a singularity of the form $\varphi(q_1, q_3)^2(\log(\varphi(q_1, q_3) + i0)^2$ near $\varphi = 0$. Since the Ω-integration (along a suitably detoured path) is over the compact set, F itself behaves as $\varphi^2(\log(\varphi + i0))^2$.

2.iii. The contribution from a coupling of a non-meromorphic part with either a meromorphic part or a C-part.

If a meromorphic part is coupled with a non-meromorphic part, the RHS of (2.ii.1) is replaced by an integral of the following form:

$$\int_{|\Omega|=1} \frac{d^4\Omega}{\Omega^2+i0} \int_0^\delta dr \int d^4p_3 \frac{1}{p_3^2-m^2+i0} \log \frac{(q_1+p_3+r\Omega)^2-m^2+i0}{(q_1+p_3)^2-m^2+i0}$$
$$\frac{1}{(p_3-q_3)^2-m^2+i0} \; \frac{1}{2(p_3-q_3)\Omega+r\Omega^2+i0}. \qquad (2.iii.1)$$

By deforming the Ω-contour, in the manner specified in ref. 2, so that $Im\Omega^2 > 0$ and $Imp_j\Omega > 0 (j = 1, 2)$ [with $p_1 = q_1 + p_3, p_2 = p_3 - q_3$], we find the singularity of this integral near $\varphi(q_1, q_3) = 0$ is $\varphi \log(\varphi + i0)$, as there is no potentially divergent factor $1/r$.

If the meromorphic part is replaced by a C-term, then the dominant singularity is given by an integral similar to (2.iii.1) but with the replacement of the residue factor $1/(2(p_3 - q_3)\Omega + r\Omega^2)$ by $1/r(p_3 - q_3)\Omega$. Hence a potentially divergent factor $1/r$ arises. But this problem is circumvented by combining the singularity originating from $\log((q_1 + p_3 + r\Omega)^2 - m^2 + i0)$ and that from $\log((q_1 + p_3)^2 - m^2 + i0)$; the results in § 4 show, with a reasoning similar to (but simpler than) that in § 2.ii, that the resulting singularity is $\varphi(\log(\varphi + i0))^2$.

3. A normalization of the function defining a Landau surface.

The purpose of this section is to prove the following lemma, which is an adaptation of the implicit function theorem (or the Weierstrass preparation theorem in the theory of holomorphic functions of several variables) to the Landau surface shifted by a vector Δ determined by photons that bridge star lines. (cf.§ 11. of Ref. 1.). Here and in what follows, $(r_1, ..., r_n)$ denotes a nested set of polar coordinates introduced in Ref. 1, §5.

Lemma 3.1 Let $\varphi(q)$ denote a defining function of the Landau surface for the triangle diagram and let $\hat{q}$ be a point on the surface. Let i be the smallest j such that j identifies a bridge photon line. (A *bridge* photon line is a photon line that has meromorphic couplings on both ends and that completes — via the rules defined below Eqn. (2)of ref. 2 — to a closed photon loop that passes along at least one star line.) Then on a sufficiently small neighborhood of q_0 and for sufficiently small $\rho_i = r_1 \cdots r_i$ there exist non-vanishing holomorphic functions $B(q, \rho_i, k'/\rho_i)$ and $C(q, k'/\rho_i)$ such that

$$\varphi(q - \Delta) = B(q, \rho_i, k'/\rho_i)(\rho_i - \varphi(q)/C(q, k'/\rho_i)) \tag{3.1}$$

holds, where k' denotes the collection of bridge lines.

Proof. Since i is the first bridge photon line, any bridge photon line k_ℓ has the form $k_\ell = \rho_i r_{i+1} \cdots r_\ell \Omega_\ell$. Hence k'/ρ_i is actually independent of ρ_i. Furthermore, as is shown at the beginning of section 5 ((5.1)),$\partial\varphi(q - \Delta)/\partial\rho_i|_{\rho_i=0} \neq 0$ holds. Hence the Weierstrass preparation theorem guarantees the local and unique existence of a non-vanishing holomorphic function $B(q, \rho_i, k'/\rho_i)$, and a holomorphic function $R(q, k'/\rho_i)$, which vanishes for $q_i = q_0$, for which the following holds:

$$\varphi(q - \Delta) = B(q, \rho_i, k'/\rho_i)(\rho_i - R(q, k'/\rho_i)) \tag{3.2}$$

Setting $\rho_i = 0$ in (3.2) we find

$$\varphi(q) = B(q, 0, k'/\rho_i)(-R(q, k'/\rho_i),$$

that is,

$$R(q, k'/\rho_i) = \varphi(q)/(-B(q, 0, k'/\rho_i)).$$

Hence by choosing $C(q, k'/\rho_i) = -B(q, 0, k'/\rho_i)$ we obtain (3.1).

4. Some auxiliary integrals.

The purpose of this section is to find an explicit form of the singularities of several integrals that we encounter in dealing with infrared problems. The simplest example of this sort is the following integral $I(t)$:

$$I(t) = \int_0^\delta [\log(r+t+i0) - \log(t+i0)]dr/r, \quad (\delta > 0).$$

In spite of the divergence factor $1/r$, $I(t)$ is well defined as a (hyper) function of t. In order to see this, it suffices to decompose $I(t)$ as

$$\int_0^{t/2} (\log(r+t) - \log t)dr/r + \int_{t/2}^\delta (\log(r+t) - \log t)dr/r$$

with $Im\ t > 0$: the well-definedness of the second integral is clear, while the fact that

$$\log(r+t) - \log t = \log(1+\frac{r}{t}) \sim \frac{r}{t}$$

holds in the domain of integration of the first integral entails its well-definedness. Furthermore $I(t)$ (thus seen to be well-defined) satisfies the following ordinary differential equation:

$$t\frac{d}{dt}I(t) = \log(t+i0) - \log(t+\delta+i0). \tag{4.1}$$

Hence $(t\frac{d}{dt})^2$ is holomorphic near $t=0$. Then it follows from the general theory of ordinary differential equations that $I(t)$ has the form

$$C_2(\log(t+i0))^2 + C_1(\log(t+i0)) + h(t), \tag{4.2}$$

where C_1 and C_2 are constants and $h(t)$ is holomorphic near $t=0$. Furthermore, by substituting (4.2) into (4.1) and comparing the coefficients of singular terms at $t=0$, we find $C_2 = 1/2$.

This computation can be generalized as follows:

Proposition 4.1. Let $J(\alpha, j; t)$ $(\alpha \neq 0, 1, 2, \cdots; j \geq 1)$ denote the following integral:

$$\int_0^{\delta=\rho_0} \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_{j-2}} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} (t+\rho_j+i0)^\alpha d\rho_j.$$

Then the singularity of $J(\alpha, j; t)$ near $t = 0$ is of the following form with some constants C_ℓ ($\ell = 0, \cdots, j-1$):

$$\begin{cases} (t+i0)^{\alpha+1}(\sum_{\ell=0}^{j-1} C_\ell(\log(t+i0))^\ell), & \text{if } \alpha \neq -1 \\ \sum_{\ell=0}^{j-1} C_\ell(\log(t+i0))^{\ell+1}, & \text{if } \alpha = -1 \end{cases}. \tag{4.3}$$

Remark 4.1. If α is a non-negative integer, the integral $J(\alpha, j; t)$ is not singular at $t = 0$.

Proof of Proposition 4.1. The well-definedness can be verified by the same method as was used for the above example $I(t)$. To find its singularity structure, we again make use of an ordinary differential equation as follows:

$$\begin{aligned}
&(t\frac{d}{dt} - (\alpha+1))J(\alpha, j; t) = \left(t\frac{d}{dt} - (\alpha+1)\right)\int_0^\delta \frac{d\rho_1}{\rho_1}\int_0^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots \\
&\int_0^{\rho_{j-2}}\frac{d\rho_{j-1}}{\rho_{j-1}}((t+\rho_{j-1}+i0)^{\alpha+1} - (t+i0)^{\alpha+1})/(\alpha+1) \\
&= \int_0^\delta \frac{d\rho_1}{\rho_1}\int_0^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots\int_0^{\rho_{j-2}}\frac{d\rho_{j-1}}{\rho_{j-1}}(t(t+\rho_{j-1}+i0)^\alpha - (t+\rho_{j-1}+i0)^{\alpha+1}) \\
&= -\int_0^\delta \frac{d\rho_1}{\rho_1}\int_0^{\rho_1}\frac{d\rho_2}{\rho_2}\cdots\int_0^{\rho_{j-2}}(t+\rho_{j-1}+i0)^\alpha d\rho_{j-1} = -J(\alpha, j-1; t)(j \geq 2).
\end{aligned}$$

Repeating this computation, we finally obtain

$$(t\frac{d}{dt} - (\alpha+1))^{j-1}J(\alpha, j; t) = (-1)^{j-1}\int_0^\delta (t+\rho_1)^\alpha d\rho_1,$$

and hence we find $(t\frac{d}{dt} - (\alpha+1))^j J(\alpha, j; t)$ is holomorphic near $t = 0$. Again, by using the general theory of ordinary differential equations, we obtain the required formula (4.3).

Remark 4.2. Although we do not need the exact values of C_ℓ's, we note that C_{j-1} in (4.3) is simply given by $(-1)^{j-2}/(j-1)!(\alpha+1)$ if $\alpha \neq -1$. In order to find this value it suffices to insert (4.3) into the recurrence relation $(td/dt - (\alpha+1))J(\alpha, j; t) = -J(\alpha, j-1; t)$ and use the trivial relation

$$\begin{aligned}
\left(t\frac{d}{dt} - (\alpha+1)\right)J(\alpha, 2; t) &= -\int_0^\delta (t+\rho_1+i0)^\alpha d\rho_1 \\
&= ((t+i0)^{\alpha+1}/(\alpha+1)) - ((t+\delta+i0)^{\alpha+1}/(\alpha+1))
\end{aligned}$$

as the starting point of the induction. Similarly C_{j-1} for $\alpha = -1$ is equal to $(-1)^j/j!$. The coefficient of the most singular term can be similarly computed explicitly for the integrals to be dealt with in subsequent propositions.

The following modification of Proposition 4.1 is often effective in actual computations.

Proposition 4.2. (i) Let $K(\alpha, j; t)$ $(\alpha \neq 0, 1, \cdots; j \geq 1)$ denote the following integral:

$$\int_0^\delta \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_j-2} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} \rho_j (t + \rho_j + i0)^\alpha d\rho_j.$$

Then its singularity near $t = 0$ is of the following form for some constants $C_\ell (\ell = 0, \cdots, j-1)$:

$$\begin{cases} (t+i0)^{\alpha+2} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^\ell \right) & \text{if } \alpha \neq -1, -2 \\ t^{\alpha+2} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^{\ell+1} \right) & \text{if } \alpha = -1 \text{ or } -2 \end{cases}. \tag{4.4}$$

(ii) Let n be a non-negative integer and let $I(n, j; t)(j \geq 1)$ denote the following integral:

$$\int_0^\delta \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_{j-2}} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} (t + \rho_j)^n \log(t + \rho_j + i0) d\rho_j.$$

Then its singularity near $t = 0$ is of the following form with some constants $C_\ell (\ell = 0, \cdots, j-1)$:

$$t^{n+1} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^{\ell+1} \right). \tag{4.5}$$

(iii) Let $\widetilde{I}(n, j; t)$(n; a non-negative integer, and $j \geq 1$) denote the following integral:

$$\int_0^\delta \frac{d\rho_1}{\rho_1} \int_0^{\rho_1} \frac{d\rho_2}{\rho_2} \cdots \int_0^{\rho_{j-2}} \frac{d\rho_{j-1}}{\rho_{j-1}} \int_0^{\rho_{j-1}} \rho_j (t + \rho_j)^n \log(t + \rho_j + i0) d\rho_j.$$

Then its singularity near $t = 0$ is of the following form with some constants $C_\ell (\ell = 0, \cdots, j-1)$:

$$t^{n+2} \left(\sum_{\ell=0}^{j-1} C_\ell (\log(t+i0))^{\ell+1} \right). \tag{4.6}$$

Proof. Since $\rho_j(t+\rho_j)^\alpha = (t+\rho_j)^{\alpha+1} - t(t+\rho_j)^\alpha$, (i) and (iii) follow respectively from Proposition 4.1 and from (ii) above. Hence it remains to prove (ii). Since

$$\frac{d^{n+1}}{dt^{n+1}}\left((t+\rho_j)^n \log(t+\rho_j+i0)\right) = \frac{n!}{t+\rho_j+i0} + P_n,$$

where P_n is a polynomial of $(t+\rho_j)$, Remark 4.1, entails that

$$\frac{d^{n+1}}{dt^{n+1}} I(n,j;t) + n! J(-1,j;t) + h(t) \tag{4.7}$$

holds with a holomorphic function $h(t)$. On the other hand, near $t = 0$ a straightforward computation shows

$$\int^t dt\ t^n (\log(t+i0))^m = \frac{t^{n+1}}{n+1} \sum_{r=0}^{m} \frac{(-1)^r m! (\log(t+i0)^{m-r}}{(m-r)!(n+1)^r} \tag{4.8}$$

holds for non-negative integers n, and $m\neg 1$. Combining (4.7) and Proposition 4.1, we use (4.8) repeatedly to find (4.5). We also note that Remark 4.2 entails $C_{j-1} = (-1)^j / j!(n+1)$ in this case.

The following proposition is a key result of this section.

Proposition 4.3. Let $I(t)$ denote the following integral (4.9), where e_j ($j = 1, 2, \cdots, n$) is a non-negative integer:

$$\int_0^\delta r_1^{e_1} dr_1 \int_0^1 r_2^{e_2} dr_2 \cdots \int_0^1 r_n^{e_n} dr_n \log(t + r_1 \cdots r_n + i0). \tag{4.9}$$

Then its singularity near $t = 0$ is a sum of finitely many terms of the form

$$Ct^N (\log(t+i0))^m \tag{4.10}$$

with a constant C and positive integers $N(\geq min \quad e_j + 1)$ and $m(\leq n)$.

Proof. First of all, let us re-scale the parameter r_1 and the variable t as follows:

$$r_1 = \delta r_1', \quad t = \delta s$$

Then I becomes

$$\delta^{-e_1-1} \int_0^1 (r_1')^{e_1} dr_1' \int_0^1 r_2^{e_2} dr_2 \cdots \int_0^1 r_n^{e_n} dr_n (\log(s + r_1' r_2 \cdots r_n) + \log \delta). \tag{4.11}$$

The contribution from $\log\delta$ in (4.11) is a finite constant. Thus we may assume from the first that $\delta = 1$. Then the roles of $r_j's$ in (4.9) are uniform, and hence we may re-number the index j so that

$$e_1 \geq e_2 \geq \cdots \geq e_n. \tag{4.12}$$

Let us introduce new variables σ_j by

$$\sigma_1 = r_1, \quad \sigma_2 = r_1 r_2, \cdots, \ \sigma_n = r_1 \cdots r_n. \tag{4.13}$$

The integral I (with $\delta = 1$) can be now expressed as

$$\int_0^1 \frac{\sigma_1^{d_1}}{\sigma_1} d\sigma_1 \int_0^{\sigma_1} \frac{\sigma_2^{d_2}}{\sigma_2} d\sigma_2 \cdots \int_0^{\sigma_{n-2}} \frac{\sigma_{n-1}^{d_{n-1}}}{\sigma_{n-1}} d\sigma_{n-1} \int_0^{\sigma_{n-1}} \sigma_n^{e_n} d\sigma_n \log(t+\sigma_n+i0), \tag{4.14}$$

where $d_j = e_j - e_{j+1}$. The number d_j is nonnegative by (4.12), and this non-negativity makes our reasoning much simpler: that is why we re-numbered the index j.

The first integration in (4.14), i.e. $\int_0^{\sigma_{n-1}} \sigma_n^{e_n} d\sigma_n \log(t+\sigma_n+i0)$, can be done in a straightforward manner: using the identity $\sigma^e = \sum_{j=0}^{e} c_j t^j (t+\sigma)^{e-j}$, where c_j is some constant, we find it is a sum of terms of the form

$$\begin{aligned} Ct^j\{(t \ + \sigma_{n-1})^{e_n-j+1} &\log(t+\sigma_{n-1}+i0) \\ &- t^{e_n-j+1}\log(t+i0)\} \end{aligned} \tag{4.15}$$

and polynomials of the form

$$C't^j\{(t+\sigma_{n-1})^{e_n-j+1} - t^{e_n-j+1}\},$$

where C and C' are some constants.

If $d_{n-1} \geq 1$, the same computation can be done for the second integration (4.14). In this case we do not need to combine the first term and the second term in (4.15). That is, we perform the integration of these terms separately. If d_{n-1} is equal to 0, we first define an integral J by

$$\int_0^1 \frac{\sigma^{d_1}}{\sigma_1} d\sigma_1 \int_0^{\sigma_1} \frac{\sigma_2^{d_2}}{\sigma_2} d\sigma_2 \cdots \int_0^{\sigma_{n-2}} \frac{d\sigma_{n-1}}{\sigma_{n-1}} \{(t+\sigma_{n-1})^\alpha \log(t+\sigma_{n-1}+i0) - t^\alpha \log(t+i0)\},$$

where $\alpha = e_n - j + 1$ is a positive integer. Then we have

$$
\begin{aligned}
(t\frac{d}{dt} &- \alpha)J(t) \\
&= \int_0^1 \frac{\sigma_1^{d_1}}{\sigma_1} d\sigma_1 \int_0^{\sigma_1} \frac{\sigma_2^{d_2}}{\sigma_2} d\sigma_2 \cdots \int_0^{\sigma_{n-2}} d\sigma_{n-1}\{-\alpha(t+\sigma_{n-1})^{\alpha-1}\log(t+\sigma_{n-1}+i0) \\
&+ ((t+\sigma_{n-1})^{\alpha-1} - t^{\alpha-1})/\sigma_{n-1}\}
\end{aligned}
$$

Since the contribution from $((t+\sigma_{n-1})^{\alpha-1} - t^{\alpha-1})/\sigma_{n-1}$ is finite and analytic in t (actually a polynomial), the main contribution to $J(t)$ is from $-\alpha(t+\sigma_{n-1})^{\alpha-1}\log(t+\sigma_{n-1}+i0)$. But this is the same integral discussed at the first step. Repeating this procedure we finally find that I has the form $\sum_{j=0}^{e_n} t^j I_j$, where I_j satisfies the following equation:

$$
\prod_{\ell=1}^{n(j)} (t\frac{d}{dt} - \alpha_\ell(j))I_j(t) = \sum_k (C_k \log(t+i0) + C'_k)t^k + A, \tag{4.17}
$$

where A is analytic at $t=0$. Here $n(j) \leq n-1$, $\alpha_\ell(j)$ is an integer $\geq e_n - j + 1$, k ranges over a finite subset of integers $\geq e_n - j + 1$, and C_k and C'_k are constants. As a solution of the equation (4.17), $I_j(t)$ [modulo a function analytic at $t=0$] is a sum of terms of the form

$$
Ct^N(\log(t+i0))^m
$$

with a constant C and integers $N \geq e_n - j + 1$ and $m \leq n$. Thus $I(t)$ consists of terms of the required form (4.10). Note that min $e_j = e_n$ by the re-numbering of the index j.

Remark 4.3. (i) If log $(t + r_1 ... r_n + i0)$ in (4.9) is replaced by $(t + r_1...r_n + i0)^\alpha$ (α: non-integer), the resulting integral is a finite sum of terms of the form

$$
Ct^{\alpha+e}(\log(t+i0))^m \tag{4.10$'$}
$$

with an integer $e \geq \min e_j + 1$ and a non-negative integer $m \leq n-1$. If $\alpha = -1$, then the condition on e is the same as above but the condition on m is replaced by $m \leq n$.

(ii) Let $a(r')$ be an analytic function of $r' = (r'_1..., r'_n)$, in a closed "cube" $C = [\epsilon, 1] \times ... \times [\epsilon, 1] (\epsilon > 0)$. Then the following integral $F(a)$ has the same

singularity as $I(t)$, or a weaker one :

$$F(a) = \int_\epsilon^1 dr'_1 ... \int_\epsilon^1 dr'_{n'} a(r') \int_0^\epsilon r_1^{e_1} dr_1 ... \int_0^\epsilon r_n^{e_n} dr_n \log(t + r'_1 ... r'_{n'} ... r_n + i0).$$

In fact, for r' in C we find

$$\begin{aligned} & \int_0^\epsilon r_1^{e_1} dr_1 ... \int_0^\epsilon r_n^{e_n} dr_n \log(t + r'_1 ... r'_m r_1 ... r_n + i0) \\ = & \int_0^\epsilon r_1^{e_1} dr_1 ... \int_0^\epsilon r_n^{e_n} dr_n \{ \log(\frac{t}{r'_1 ... r'_m} + r_1 ... r_n + i0) \\ & + \log(r'_1 ... r'_m) \}. \end{aligned}$$

Since the contribution from $\log(r'_1 ... r'_m)$ to $F(a)$ is an analytic function, it suffices to consider the contribution from $\log(\frac{t}{r'_1 ... r'_{n'}} + r_1 ... r_n + i0)$. Proposition 4.3 then tells us that it is a sum of terms of the form

$$\int_\epsilon^1 dr'_1 ... \int_\epsilon^1 dr'_{n'} a(r') \frac{t^N}{(r'_1 ... r'_{n'})^N} (\log(t + i0) - \log(r'_i ... r'_{n'}))^m.$$

Hence the singularity of $F(a)$ near $t = 0$ is a sum of terms of the form (4.10). Note that the effect of changing the upper end-point of the integral in (4.9) to ϵ is absorbed by the harmless change of scaling in r variables and t variable (as was employed at the beginning of the proof of Proposition 4.3), on the condition that ϵ is a fixed positive constant.

To generalize Proposition 4.3 to the form needed in section 6 we prepare the following Lemma:

Lemma 4.1. Let $L_m(t; a)(n = 1, 2, ...; a$ a strictly positive constant), denote the following integral:

$$\int_0^a \frac{(\log w)^m}{t + w + i0} dw.$$

Then the singularity structure of L_m near $t = 0$ is as follows:

$$L_m = \sum_{j=1}^{m+1} C_j (\log(t + i0))^j + h(t) \tag{4.18}$$

where $C_j (j = 1, ..., m + 1)$ are some a-dependent constants and $h(t)$ is an a-dependent holomorphic function near $t = 0$.

Proof. Since $(\log w)^m \theta(w) \theta(a - w) (a > 0)$ is well-defined as a product of locally summable functions, the convolution-type integral $L_m(t; a)$ is well-defined, and

it is a boundary value of a holomorphic function on $\{\operatorname{Im} t > 0\}$ near $t = 0$. To find out its explicit form (4.18), we first apply an integration by parts:

$$\begin{aligned}
L_m \; &= \lim_{\kappa \downarrow 0} \Bigg\{ - \int_\kappa^a \frac{m(\log(w+i0))^{m-1}}{w+i0} \log(t+w+i0) dw \\
&+ (\log a)^m \log(a+t+i0) - (\log(\kappa+i0))^m \log(t+\kappa+i0) \Bigg\} \\
&= \lim_{\kappa \downarrow 0} \Bigg\{ - \int_\kappa^a \frac{m(\log(w+i0))^{m-1}}{w+i0} \log\left(\frac{t+w+i0}{t+i0}\right) dw \\
&- \left(\int_\kappa^a \frac{m(\log(w+i0))^{m-1}}{w+i0} dw \right) \log(t+i0) - (\log(\kappa+i0))^m \log(t+\kappa+i0) \} \\
&+ (\log a)^m \log(a+t+i0) \\
&= \lim_{\kappa \downarrow 0} \Bigg\{ - \int_\kappa^a \frac{m(\log(w+i0))^{m-1}}{w+i0} \log\left(\frac{t+w+i0}{t+i0}\right) dw \\
&- (\log(\kappa+i0))^m \log \frac{t+\kappa+i0}{t+i0} \Bigg\} + (\log a)^m \log \frac{t+a+i0}{t+i0}
\end{aligned} \tag{4.19}$$

Let us note that, if $\operatorname{Im} t > 0$,

$$\begin{aligned}
&(\log(\kappa+i0))^m \log \frac{t+\kappa+i0}{t+i0} \\
&= (\log(\kappa+i0))^m \left(\frac{\kappa+i0}{t+i0} - \frac{1}{2} \left(\frac{\kappa+i0}{t+i0} \right)^2 + \ldots \right) \longrightarrow 0
\end{aligned}$$

as $\kappa \downarrow 0$. Hence we obtain

$$L_m = M_m - (\log a)^m \log(t+i0) + (\log a)^m \log(t+a+i0), \tag{4.20}$$

where

$$M_m = \lim_{\kappa \downarrow 0} \left(- \int_\kappa^a \frac{m(\log(w+i0))^{m-1}}{w+i0} \log \; \frac{t+w+i0}{t+i0} dw \right). \tag{4.21}$$

Since $(\log\,(w+i0))^{m-1}$ is locally summable, the reasoning used to verify the well-definedness of the integral $\displaystyle\int_0^\delta \frac{dr}{r} \log \frac{t+r+i0}{t+i0}$ (cf. the beginning of this appendix) is applicable also to M_m. To find out the explicit form of M_m, let us first note

$$M_1 = C_2(\log(t+i0))^2 + C_1(\log(t+i0)) + h(t)$$

holds near $t = 0$ for some constants C_1, C_2 and some holomorphic function $h(t)$. (Cf. (4.2)). Thus we can verify (4.18) for $n = 1$. For $n > 1$, we use mathematical induction: Let us suppose (4.18) is verified for $1 \le m \le m_0$. Since

$$t\frac{d}{dt}M_{m_0+1} = \int_0^a \frac{(m_0+1)(\log(w+i0))^{m_0}}{t+w+i0}dw = (m_0+1)L_{m_0}, \tag{4.22}$$

using the induction hypothesis, we find that $(t\frac{d}{dt})^{m_0+1}L_{m_0}$ is holomorphic near $t = 0$. This means that $(t\frac{d}{dt})^{m_0+2}M_{m_0+1}$ is holomorphic near $t = 0$. Otherwise stated,

$$M_{m_0+1} = \sum_{j=1}^{m_0+2} \widetilde{C}_j(\log(t+i0))^j + \widetilde{h}(t)$$

holds near $t = 0$ for some constants $\widetilde{C}_j(j = 1, ..., m_0+2)$ and some holomorphic function $\widetilde{h}(t)$. Therefore (4.20) implies that (4.18) is true for $m = m_0+1$. Thus the induction proceeds.

Proposition 4.4. (i) Let $K_{n,m}(t)(n, m = 1, 2, 3, ...)$ denote the following integral (with $\delta > 0$) :

$$\int_0^\delta (\log r_0)^m dr_0 \int_0^1 ... \int_0^1 \prod_{j=1}^n dr_j (t + r_0 r_1 ... r_n + i0)^{-1}. \tag{4.23}$$

Then the singularity structure of $K_{n,m}(t)$ near $t = 0$ is as follows:

$$K_{n,m}(t) = \sum_{j=1}^{n+m+1} C_j(\log(t+i0))^j + h(t), \tag{4.24}$$

where $C_j(j = 1, ..., n+m+1)$ are some constants and $h(t)$ is some holomorphic function near $t = 0$.

(ii) Let $J_{n,m}(t)(n, m = 1, 2, 3, ...)$ denote the following integral:

$$\int_0^\delta (\log r_0)^m dr_0 \int_0^1 ... \int_0^1 \prod_{j=1}^n dr_j \log(t + r_0 r_1 ... r_n + i0). \tag{4.25}$$

Then the singularity structure of $J_{n,m}(t)$ near $t = 0$ is as follows:

$$J_{n,m}(t) = \sum_{j=1}^{n+m+1} C_j t(\log(t+i0))^j + h(t), \tag{4.26}$$

where $C_j(j = 1, ..., n+m+1)$ are some constants and $h(t)$ is some holomorphic function near $t = 0$.

Proof. (i) Let $\rho_j (j = 0, 1, ..., n)$ denote $\prod_{i=0}^{j} r_i$. Then $K_{n,m}$ assumes the following form:

$$\int_0^{\delta} \frac{(\log \rho_0)^m}{\rho_0} d\rho_0 \int_0^{\rho_0} \frac{d\rho_1}{\rho_1} ... \int_0^{\rho_{n-2}} \frac{d\rho_{n-1}}{\rho_{n-1}} \int_0^{\rho_{n-1}} d\rho_n (t + \rho_n + i0)^{-1}.$$

Hence we find

$$(-t\frac{d}{dt})^n K_{n,m}(t) = \int_0^{\delta} \frac{(\log \rho_0)^m}{t + \rho_0 + i0} d\rho_0.$$

Therefore Lemma 4.1 shows that $(-t\frac{d}{dt})^{n+m+1} K_{n,m}(t)$ is holomorphic near $t = 0$. This entails (4.24).

(ii) Since $\frac{d}{dt} J_{n,m} = K_{n,m}$, the result (i) entails

$$\frac{d}{dt} J_n = \sum_{j=1}^{n+m+1} C_j (\log(t + i0))^j + h(t) \tag{4.27}$$

holds for some constants C_j and a holomorphic function $h(t)$. Hence, by integrating both sides of (4.27), we find (4.26). Here we have used a formula

$$\int^t (\log t)^N dt = t \left(\sum_{\ell=0}^{N} (-1)^{\ell} \frac{N!}{(N-\ell)!} (\log t)^{N-\ell} \right).$$

The following generalization of Proposition 4.4 is used in section 6

Proposition 4.5. Let $L_{n,m}(t) (n, m = 1, 2, 3, ...)$ denote the following integral (with $\delta_0 > 0$), where $e_j (j = 0, 1, ..., n)$ is a non-negative integer:

$$\int_0^{\delta_0} (\log r_0)^m r_0^{e_0} dr_0 \int_0^1 ... \int_0^1 \prod_{j=1}^{n} r_j^{e_j} dr_j \log(t + r_0 r_1 r_2 ... r_n + i0).$$

Then the singular part of $L_{n,m}(t)$ near $t = 0$ is a finite sum of terms of the following form:

$$C_{N,p}\ t^N (\log(t + i0))^p, \tag{4.28}$$

where $C_{N,p}$ is a constant, N is a non-negative integer ($\geq \min_{0 \leq j \leq n} e_j + 1$) and p is a positive integer ($\leq n + m + 1$).

Proof. Making use of the scaling transformation of r_0 and t as in the proof of Proposition 4.3, we may assume without loss of generality that $\delta_0 = 1$. Furthermore, as the role of variables $r_j (j = 1, ..., n)$ is uniform, we may assume, by re-labelling of the variables $r_j (j = 1, ..., n)$, that $e_1 \geq e_2 \geq ... \geq e_n$. If $e_0 \geq e_1$

then the method used in the proof of Proposition 4.3, supplemented by Lemma 4.1, establishes the required result. However, this condition cannot be expected to hold in general, and hence we must generalize. Introducing the new variables $\sigma_j = r_0 r_1 .. r_j$ $(j = 0, 1, ..., n)$ we find

$$\begin{aligned} L_{n,m}(t) &= \int_0^1 (\log \sigma_0)^m \sigma_0^{d_0 - 1} d\sigma_0 \int_0^{\sigma_0} \sigma_1^{d_1 - 1} d\sigma_1 ... \int_0^{\sigma_{n-2}} \sigma_{n-1}^{d_{n-1} - 1} d\sigma_{n-1} \\ &\times \int_0^{\sigma_{n-1}} \sigma_n^{e_n} \log(t + \sigma_n + i0) d\sigma_n, \end{aligned}$$

where $d_j = e_j - e_{j+1}$. As noted above, the proof is finished if $d_0 \geq 0$. Let us consider the case $d_0 < 0$. We then use mathematical induction on m. When $m = 1$, we use the following:

$$\begin{aligned} &\frac{d}{d\sigma_0} ((\log \sigma_0) \sigma_0^{d_0} F(\sigma_0, t)) \\ &\quad = d_0 (\log \sigma_0) \sigma_0^{d_0 - 1} F(\sigma_0, t) + \sigma_0^{d_0 - 1} F(\sigma_0, t) \\ &\quad + (\log \sigma_0) \sigma_0^{d_0} \frac{\partial F(\sigma_0, t)}{\partial \sigma_0}. \end{aligned} \tag{4.29}$$

Choosing

$$\int_0^{\sigma_0} \sigma_1^{d_1 - 1} d\sigma_1 ... \int_0^{\sigma_{n-2}} \sigma_{n-1}^{d_{n-1} - 1} d\sigma_{n-1} \int_0^{\sigma_{n-1}} \sigma_n^{e_n} \log(t + \sigma_n + i0) d\sigma_n$$

as $F(\sigma_0, t)$, we obtain

$$\begin{aligned} d_0 L_{n,1} &= (\log \sigma_0) \sigma_0^{d_0} F(\sigma_0, t)|_{\sigma_0 = 1} - \lim_{\sigma_0 \downarrow 0} ((\log \sigma_0) \sigma_0^{d_0} F(\sigma_0, t)) \\ &- \int_0^1 \sigma_0^{d_0 - 1} F(\sigma_0, t) d\sigma_0 - \int_0^1 (\log \sigma_0) \sigma_0^{e_0 - e_1} \sigma_0^{e_1 - e_2 - 1} \\ &\times \int_0^{\sigma_0} \sigma_2^{d_2 - 1} d\sigma_2 ... \int_0^{\sigma_{n-1}} \sigma_n^{e_n} \log(t + \sigma_n + i0) d\sigma_n. \end{aligned} \tag{4.30}$$

For notational convenience, let $A_j (j = 1, 2, 3, 4)$ denote the j-th term in RHS of (4.30). Since $F(1, t)$ is a well-defined integral (cf. Proposition 4.3), A_1 vanishes because of the trivial fact $\log 1 = 0$. To confirm that A_2 also vanishes, we note that

$$\left| \frac{1}{t + \sigma_n} \right| \leq C_\epsilon$$

holds if $\operatorname{Im} t \geq \epsilon > 0$ and σ_n is real. Then we find, for $\sigma_0 \geq 0$,

$$
\begin{aligned}
|F(\sigma_0, t)| &\leq C_\epsilon \int_0^{\sigma_0} \sigma_1^{d_1 - 1} d\sigma_1 ... \int_0^{\sigma_{n-1}} \sigma_n^{e_n} d\sigma_n \\
&= \frac{C_\epsilon \sigma_0^{e_1+1}}{\prod_{j=1}^n (e_j + 1)}.
\end{aligned}
$$

Therefore

$$
|A_2| \leq C_\epsilon \lim_{\sigma_0 \downarrow 0} \frac{(\log \sigma_0) \sigma_0^{e_0+1}}{\prod_{j=1}^n (e_j + 1)} = 0.
$$

Since ϵ is an arbitrary positive number, this means that A_2 vanishes.

The term A_3 has the same structure as the integral discussed in Proposition 4.3, and hence its singular part is a sum of terms of the form (4.28). Note that we can re-label all variables including r_0 if we go back to r-variables from σ-variables in the integral A_3; the factor $\log \sigma_0$ has disappeared in A_3.

Finally let us study A_4. As it has the form $L_{n-1,1}$, we can apply the above procedure to it. Repeating this procedure , we eventually end up with one of the following two integrals (i) or (ii), together with terms of the form (4.28):

(i) $L_{n',1} (n' < n)$ with $d_0 \geq 0$

(ii) $\int_0^1 (\log \sigma_0) \sigma_0^{e_0+1} \log(t + \sigma_0 + i0) d\sigma_0$.

By using Lemma 4.1 together with the method of the proof of Proposition 4.3, we can verify that the singular part of either of them is a sum of terms of the form (4.28).

Thus the proof is finished if $m = 1$. Let us consider next the case $m \geq 2$. We then use

$$
\begin{aligned}
&\frac{d}{d\sigma_0} \left((\log \sigma_0)^m \sigma_0^{d_0} F(\sigma_0, t) \right) \\
&\qquad = d_0 (\log \sigma_0)^m \sigma_0^{d_0 - 1} F(\sigma_0, t) + m (\log \sigma_0)^{m-1} \sigma_0^{d_0 - 1} F(\sigma_0, t) \\
&\qquad + (\log \sigma_0)^m \sigma_0^{d_0} \frac{\partial F(\sigma_0, t)}{\partial \sigma_0}.
\end{aligned} \tag{4.31}
$$

As before, we concentrate our attention on the case $d_0 < 0$. Choosing as $F(\sigma_0, t)$ the same integral as was used when $m = 1$, we find that the same reasoning as before applies to the contribution from the LHS of (4.31) and the third term on the RHS of (4.31). When integrated over [0, 1] (with respect to σ_0), the

second term on the RHS of (4.31) turns out to be $mL_{n,m-1}$. Thus the induction proceeds, completing the proof.

5. Weakness of the singularity in the general non-separable meromorphic case.

To confirm the weakness of the singularity in the non-separable meromorphic case we first need to verify

$$\frac{\partial}{\partial \rho_i} \varphi(q - \Delta)|_{\varphi=0} \neq 0, \tag{5.1}$$

where $\rho_i = r_1 \cdots r_i$, with i being the index labelling the first *bridge* line; i.e., i is the smallest j such that the photon line j has a meromorphic coupling on both ends, and completes to a closed loop — constructed according to the rules specified below Eq.(2) in Ref. 2 — that flows along at least one $*$-segment. The k_i-dependent vector Δ is chosen so that at $\varphi = 0$ the pole factor associated with each $*$-segment can be evaluated at the critical point $p_s(q - \Delta)$ $(s = 1, 2, 3)$, defined below Fig. 1 in Ref. 2, with $q = (q_1, q_2, q_3)$ the set of external variables defined there.

The vector Δ is constructed in the following way. Introduce for each bridge line i an open flow line $L(k_i)$ that passes along this photon line i, but along no other photon line, and along no $*$-segment. Instead, the flow line $L(k_i)$ *enters* the diagram at one of the three vertices v_i and *leaves* at another. Specifically, let e be an end-point of the photon line i, and let s be the side of the triangle on which e lies. This point e separates s into two connected components, s^0 and s^*, where s^* is the part of s that contains the $*$-segment. Run $L(k_i)$ along the component s^0. At the end-point of s^0 that coincides with a vertex v_i of the triangle diagram, run $L(k_i)$ out along the external line q_j $(j = 1, 2$ or $3)$. Do the same for the other end-point of the line i. Include on $L(k_i)$ also the segment i itself. This produces a continuous flow line. Orient it so that it agrees with the orientation of the line i. This oriented line is the flow line $L(k_i)$. Then for each external line q_j along which $L(k_i)$ runs add to the vector q_j either $+k_i$ or $-k_i$ according to whether the orientation of $L(k_i)$ is the same as, or opposite to, the orientation of the external line q_j along which $L(k_i)$ runs. Sum up the contributions from all of the bridge lines. This shift in $q = (q_1, q_2, q_3)$ is the vector Δ.

The function of interest has the form

$$
\begin{aligned}
F(q) &= \prod_{j=1}^{n} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \prod_{j=i+1}^{n} \int_0^1 r_j^{e_j} dr_j \prod_{j=1}^{i} \int_0^1 r_j^{e_j} dr_j \\
&\quad \times A(q, \Omega, r) \log \varphi(q - \Delta),
\end{aligned}
\tag{5.2}
$$

where

$$
D \log \varphi(q - \Delta) + E = \int_{p \approx \widehat{p}} d^4 p \prod_{s=1}^{3} \frac{1}{p_s^2 - m^2 + i0}, \tag{5.3}
$$

and A, D, and E are holomorphic.

Here

$$
\begin{aligned}
p_1 &= p + q_1 + \sum_{m=i}^{n} \epsilon_{1m} k_m, \\
p_2 &= p - q_3 + \sum_{m=i}^{n} \epsilon_{2m} k_m,
\end{aligned}
\tag{5.4}
$$

and

$$
p_3 = p + \sum_{m=i}^{n} \epsilon_{3m} k_m,
$$

with each ϵ_{sm} either zero or one.

We are interested in the singularity of this function at the point $\widehat{q}$ on $\varphi(q) = 0$. This singularity comes from the p-space point $\widehat{p} = p(\widehat{q})$, and we can consider the p-space domain of integration to be some small neighborhood of $\widehat{p}$. Similarly, the domain in (r, Ω) is confined to a region R in which the following conditions hold:

$$
\begin{aligned}
(\widehat{p} + \widehat{q}_1) \cdot \sum_{m=1}^{n} \epsilon_{1m} k_m / \rho_i &\approx i\epsilon_1 \\
(\widehat{p} - \widehat{q}_3) \cdot \sum_{m=1}^{n} \epsilon_{2m} k_m / \rho_i &\approx i\epsilon_2 \\
\widehat{p} \cdot \sum_{m=1}^{n} \epsilon_{3m} k_m / \rho_i &\approx i\epsilon_3 .
\end{aligned}
\tag{5.5}
$$

That is, the real parts of the three denominators in (5.3) are close to zero, and the imaginary parts are positive: $\epsilon_s \geq 0 (s = 1, 2, 3); \sum \epsilon_s > 0$. It was shown in Ref. 2 that the contours can be distorted in a way such that (5.5) holds in a neighborhood of the points contributing to the singularity at $\widehat{q}$.

Note that all of the k_m that contribute to (5.5) belong to bridge lines, and hence have a factor ρ_i. Thus none of the $r_j (j \leq i)$ enter into (5.5). Hence the region R is independent of the variables $r_j (j \leq i)$.

The quantity $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ is added to $q = (q_1, q_2, q_3)$, and it satisfies, in analogy to $\sum q_i = 0$, the condition $\sum \Delta_i = 0$. This trivector Δ is a sum of terms, one for each bridge line. For each bridge line j the corresponding term in Δ is proportional to k_j. If line j bridges over (only) the star line on side $s = 1$ then the contribution to Δ is $(-k_j, k_j, 0)$. If line j bridges over (only) the star line on side $s = 2$ then the contribution to Δ is $(0, -k_j, k_j)$. If the line j bridge (only) over the star line on side $s = 3$ then the contribution to Δ is $(k_j, 0, -k_j)$.

The gradient of $\varphi(q)$ is also a trivector. The condition $\sum q_i = 0$ in q space means that the gradient (which is in the dual space) is defined modulo translations: $\Delta_i \to \Delta_i + X$, all i. Thus one can take $\nabla\varphi$ to have a null second component. Then at the point $\widehat{q}$ of interest the gradient has the form[3]

$$\nabla\varphi = (\alpha_1 \widehat{p}_1, 0, -\alpha_2 \widehat{p}_2), \tag{5.6}$$

provided the sign and normalization of φ are appropriately defined. Hence the quantity on the left-hand side of (5.1) is, at $q - \Delta = \widehat{q}$,

$$\begin{aligned} \frac{\partial\varphi(q-\Delta)}{\partial\rho_i} &= -\nabla\varphi \cdot \frac{\partial\Delta}{\partial\rho_i} \\ &= \sum_{s=1}^{3} \sum_{m=i}^{n} \alpha_s \widehat{p}_s \epsilon_{sm} k_m / \rho_i \end{aligned} \tag{5.7}$$

which, according to (5.5), is nonzero, as claimed in (5.1). Use has been made here of the Landau equation $\sum \alpha_s \widehat{p}_s = 0$.

Using (5.1) we now employ the result in section 3 to normalize the defining function φ of the Landau surface so that we may apply Proposition 4.3 of Section 4 to the integral F in question. It follows from Lemma 3.1 in Appendix 3 that the following normalization holds on a neighborhood of the point in question:

$$\varphi(q - \Delta) = B(q, \rho_i, k'/\rho_i)\ (\rho_i - \varphi(q)/C(q, k'/\rho_i)), \tag{5.8}$$

where B and C are different from 0 at any point in question, and k' denotes the totality of the bridge lines k_j. Note that each bridge k_j contains a factor ρ_i and

that k'/ρ_i is independent of ρ_i. Let us now apply Proposition 4.3 in section 4 to the following integral I:

$$I = \int_0^\delta r_1^{e_1} dr_1 \int_0^1 r_2^{e_2} dr_2 \cdots \int_0^1 r_i^{e_1} dr_i \log(\rho_i - \varphi/C). \tag{5.9}$$

Then we find [modulo a function analytic at $\varphi = 0$]

$$I = E(q, k'/\rho_i)\ (\varphi(q)/C(q, l'/\rho_i))^N \ \Big(\sum_{j=0}^{i} a_j(q, k'/\rho_i) \times (\log(\varphi(q)/C(q, k'/\rho_i)))^j \Big) \tag{5.10}$$

with $N \geq 1$, and E and a_j being holomorphic in their arguments, and, in particular, in the r_j's ($j > i$).

The function A in (5.2) is holomorphic. This factor has no important effect on the result: it can be incorporated by using Remark 4.3(ii) of section 4.

6. Computation for the Nonmeromorphic Case

The computation in the nonmeromorphic case is similar to the computation for the meromorphic case described in section 5 with the help of section 3. Let the special index i be now the smallest integer such that photon line i is either a bridge line or a photon line with a nonmeromorphic coupling on at least one end.

If line i is a bridge line (and hence, by definition, has a meromorphic coupling on each end, and bridges across a $*$-segment) then the argument used for the meromorphic case continues to work. This is because the condition (5.1) of section 5 continues to hold, and each variable λ_j associated with a nonmeromorphic coupling acts just like a variable $r_j (j > i)$ of sections 3 and 5.

If, on the other hand, the index i labels a line with a nonmeromorphic coupling on at least one end then (5.1) may fail, because in this case the variable k_i may enter into Δ only in the form $\lambda_i k_i$ (or $\lambda_i' k_i$). For example, if the photon line i runs between two different sides, s and s', and has a nonmeromorphic coupling on both ends then, according to (10.8b) of ref. 1, the vector k_i enters into Δ only in the combinations $\lambda_i k_i$ or $\lambda_i' k_i$, where λ_i and λ_i' are the variables associated with the two different nonmeromorphic couplings of line i. Hence the derivative on Δ occurring in (5.7) will introduce a factor λ_i or λ_i' into each k_i-dependent contribution to (5.7). Since λ_i and λ_i' vanish in the domain of integration, and all other contributions have factors $r_j (j > i)$, which can vanish, the property (5.1) can fail.

Similarly, if only one end of line i is coupled nonmeromophically, say into the side s, but the closed loop i does not pass through the star line for either of the other two sides $s' \neq s$, then again (5.1) can fail, for essentially the same reason.

These failures of (5.1) cannot be avoided by simply using $\rho_i' = \lambda_i \rho_i$ (or $\lambda_i' \rho_i$) in place of ρ_i, because the condition in (3.1) on k'/ρ_i fails if ρ_i is replaced by ρ_i'.

In this section the "self-energy" photons that couple nonmeromorphically on both ends onto the same side s will be ignored: they are treated in section 7.

To deal with the new cases we introduce the set of variables $x_j (j \epsilon J)$ to represent both the $r_j (j > i)$, and also the occurring variables $\lambda_j (j \geq i)$ and $\lambda_j' (j \geq i)$. This set $x_j (j \epsilon J)$ replaces the set $r_j (j > i)$ that occurs in the argu-

ments of section 3 and 5.

Using the evaluation (5.6) for the constant gradient vector $\nabla\varphi$ we define a new variable

$$\begin{aligned} \rho &= -\nabla\varphi \cdot \Delta \\ &= \sum_{s=1}^{3} \sum_{m=i}^{n} \alpha_s \widehat{p}_s \epsilon'_{sm} k_m, \end{aligned} \tag{6.6}$$

where the reasoning leading to (5.7) has been used. However, ϵ'_{sm} can be 0, 1, λ_m, or λ'_m, with the latter two possibilities coming from the possible nonmeromorphic couplings.

In the case under consideration the photon line i has a nonmeromorphic coupling on one or both ends. If this line i has a nonmeromorphic coupling on only one end, and the ϵ'_{si} associated with the other end is 1, then (5.1) again holds, and the method used in the meromorphic case again works. In the remaining, cases (namely those for which $\epsilon'_{si} \neq 1$ for all s) the function $r_0 = \rho/\rho_i$ has a term $\lambda_i p_s \Omega_i$ (or $\lambda'_i p_s \Omega_i$) and no other dependence on λ_i (or λ'_i). Hence the variable r_0 may be introduced as a new variable, replacing λ_i (or λ'_i), provided the associated coefficient $p_s \cdot \Omega_i$ is nonzero.

The arguments of Ref. 2, slightly extended to include the λ_j, show that $\widehat{p}_s \cdot \Omega_i$ can be taken to be nonzero near points in the integration domain that lead to a singularity of the integral at $\widehat{q}$. Hence the transformation to the new set of variables (with r_0 replacing λ_i or λ'_i) is a holomorphic transformation: all analyticity properties are maintained.

The derivative at $(q - \Delta) = \widehat{q}$ of $\varphi(q - \Delta)$ with respect to ρ is

$$\begin{aligned} \frac{\partial\varphi(q-\Delta)}{\partial\rho} &= \nabla\varphi \cdot \frac{\partial(-\Delta)}{\partial\rho} \\ &= \frac{\partial(-\nabla\varphi\cdot\Delta)}{\partial\rho} \\ &= \frac{\partial\rho}{\partial\rho} = 1. \end{aligned} \tag{6.7}$$

Thus (5.1) is now satisfied (with $\rho = r_0 r_1 ... r_i$ in place of $\rho_i = r_1 r_2 ... r_i$), and we can use the method of sections 3 and 5.

The function $F(q)$ of (5.2) now takes the form

$$F(q) = \prod_{j=1}^{n} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \prod_{j=1}^{i} \int_0^1 r_j^{e_j} dr_j \int dr_0 G(q, r, r_0), \tag{6.8}$$

where

$$\begin{aligned} G(q, r, r_0) &= \prod_{j \epsilon J} \int_0^1 x_j^{e_j} dx_j \delta(r_0 + \rho_j^{-1} \nabla\varphi \cdot \Delta) \\ &\quad \times A(q, \Omega, r, x) \log \varphi(q - \Delta), \end{aligned} \tag{6.9}$$

and $\log \varphi(q - \Delta)$ is defined in (5.3) and (5.4), but with the ϵ'_{sm} in place of the ϵ_{sm}. Notice that the $\int dr_0$ can be cancelled by the δ function to give the generalization of (5.2) engendered by the action of the nonmeromorphic-part operators of (10.8b) in Ref. 1.

The expression for G given in (6.9) is well defined only for real ∇ (i.e., only for real $k_j (j \geq i)$). A more general definition is this: (1), leave the $\int dr_0$ and δ function out of (6.8) and (6.9); (2), change the variable λ_i (or λ'_i) to r_0; (3), replace the $\int d\lambda_i$ (or $\int d\lambda'_i$) by $\int dr_0$; (4), identify G as the integrand of this integral over dr_0.

Near the point $\widehat{q}$ one can write

$$\begin{aligned} \varphi(q - \Delta) &\cong \varphi(q) - \nabla\varphi \cdot \Delta \\ &= \varphi(q) + \rho. \end{aligned} \tag{6.10}$$

Insertion of (6.10) into (6.8) and (6.9) gives

$$\begin{aligned} F(q) &= \prod_{j=1}^{n} \int_{\Omega_j \widetilde{\Omega}_j = 1} d\Omega_j \prod_{j=1}^{i} \int_0^1 r_j^{e_j} dr_j \int dr_0 \log(\varphi(q) + \rho) \\ &\quad \times f(r_0, r, \Omega, q), \end{aligned} \tag{6.11}$$

where

$$f(r_0, r, \Omega, q) = \prod_{j \epsilon J} \int_0^1 x_j^{e_j} dx_j \delta(r_0 + \nabla\varphi \cdot \widetilde{\Delta}) A(q, \Omega, r, x), \tag{6.12}$$

and $\widetilde{\Delta} \equiv \Delta / \rho_i$.

Equation (6.11) exhibits the smearing of the $\log \varphi(q)$ singularity. If $f(r_0, r, \Omega, q)$ were to have a δ function singularity at $r_0 = 0$ then the expression (6.12) would

yield a singularity of the form log $\varphi(q)$. But if f has only a milder singularity at $r_0 = 0$ then $F(q)$ will have a weaker singularity at $\varphi(q) = 0$.

Let us examine, then, the form of $f(r_0, r, \Omega, q)$. Let the particular x_j that is λ_i be called simply λ. Then $\nabla\varphi \cdot \widetilde{\Delta}$ will be $(a\lambda + P)$, where P is a sum of terms each of which is a coefficient of the form $p_s(q)\Omega_j$ times a product $r_{i+1}r_{i+2}...r_j$, or $r_{i+1}r_{i+2}...r_j\lambda_j$, or $r_{i+1}r_{i+2}...r_j\lambda'_j$. Eventually the coefficients $p_s(q)\Omega_j$ will be shifted to nonzero complex numbers. But we shall evaluate the integrals first at points where each $p_s(q)\Omega_j = 1$, $a = A = 1$, and each $e_j = 0$.

Consider first, then, for $0 < r_0 < 1$, the simple example

$$f(r_0) = \int_0^1 d\lambda \int_0^1 dx_1 \int_0^1 dx_2 \; \delta(r_0 - \lambda - x_1 x_2). \tag{6.13}$$

Using the δ function to eliminate the $\int d\lambda$ we obtain (with Θ the Heaviside function)

$$\begin{aligned} f(r_0) \; &= \int_0^1 dx_1 \int_0^1 dx_2 \; \Theta(r_0 - x_1 x_2) \\ &= \int_0^1 dx_1 \int_0^{r_0/x_1} dx_2 \; \Theta(1 - \frac{r_0}{x_1}) \\ &+ \int_0^1 dx_1 \int_0^1 dx_2 \; \Theta(\frac{r_0}{x_1} - 1) \\ &= r_0 \int_{r_0}^1 \frac{dx_1}{x_1} + \int_0^{r_0} dx_1 \\ &= r_0(-\log r_0 + 1). \end{aligned} \tag{6.14}$$

Thus in this case the singularity of the function $f(r_0)$ is much weaker than $\delta(r_0)$: $f(r_0)$ is bounded and tends to zero as r_0 tends to zero.

The general form of $f(r_0)$ is

$$f(r_0) = \prod_{j \epsilon J} \int_0^1 dx_j \; \Theta(r_0 - P), \tag{6.15}$$

where P is as defined above. One sees immediately that $f(r_0)$ is bounded, and tends to zero with r_0.

To begin the study of the general form of $f(r_0)$ let us consider a case slightly more complicated than (6.14):

$$f(r_0) \; = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \; \Theta(r_0 - x_1 x_2 x_3)$$

$$
\begin{aligned}
&= \int_0^1 dx_1 \int_0^1 dh \ \Theta(r_0 - x_1 h) \\
&\times \int_0^1 dx_2 \int_0^1 dx_3 \ \delta(h - x_2 x_3) \\
&= \int_0^1 dx_1 \int_0^1 dh \ \Theta(r_0 - x_1 h) H(h), \qquad (6.16)
\end{aligned}
$$

where (for $0 < h < 1$)

$$
\begin{aligned}
H(h) \ &= \int_0^1 dx_2 \int_0^1 dx_3 \ \delta(h - x_2 x_3) \\
&= \int_0^1 \frac{dx_2}{x_2} \ \Theta(1 - \frac{h}{x_2}) \\
&= \int_h^1 \frac{dx_2}{x_2} = (-\log h). \qquad (6.17)
\end{aligned}
$$

Notice that the last line of (6.16) has the same form as the first line of (6.14), but with a different function H. Substituting the function $H(h)$ from (6.17) into (6.16) one obtains, for $0 < r_0 < 1$,

$$
\begin{aligned}
f(r_0) \ &= \int_0^1 dx_1 \int_0^1 dh \ \Theta(\frac{r_0}{x_1} - h)(-\log h) \\
&= \int_{r_0}^1 dx_1 \int_0^{r_0/x_1} dh(-\log h) \\
&+ \int_0^{r_0} dx_1 \int_0^1 dh(-\log h) \\
&= \sum_{n,m} C_{nm} (r_0)^n (\log r_0)^m, \qquad (6.18)
\end{aligned}
$$

where only a finite number of the constant coefficients C_{nm} are nonzero.

A function of one variable x having, in $0 < x < 1$, the form $\sum C_{nm} x^n (\log x)^m$, and bounded in $0 \leq x \leq 1$, with some finite number of nonzero coefficients C_{nm}, will be said to have form F. Thus the functions $f(r_0)$ specified in (6.14) and (6.16) both have form F.

In fact, the general function $f(r_0)$ of the form specified in (6.15) has form F. To see this note first that if we replace the factor $H(h) = (-\log h)$ in (6.16) by any function $H(h)$ of form F then $f(r_0)$ has form F:

$$
f(r_0) \ = \int_{r_0}^1 dx_1 \int_0^{r_0/x_1} dh \ H(h)
$$

$$
\begin{aligned}
&+\int_0^{r_0} dx_1 \int_0^1 dh\ H(h) \\
&= r_0 \int_{r_0}^1 \frac{dx'}{(x')^2} \int_0^{x'} dh\ H(h) \\
&+ r_0 \int_0^1 dh\ H(h)
\end{aligned}
\tag{6.19}
$$

Then (4.8) and Lemma 4.1 give the result that if $H(h)$ has form F then $f(r_0)$ has form F. (Note that every term in $\int_0^{x'} dhH(h)$ has a factor x', and hence the denominator $(x')^2$ is reduced to x'.) So our problem is to show that $f(r_0)$ can be reduced to the form (6.16) with $H(h)$ having the form F.

To show this let $I(g)$ be some function of form F and consider the integral operator H_h defined by

$$
H_h[I(g)] = \int_0^1 dx \int_0^1 dg\ \delta(h - xg) I(g). \tag{6.20}
$$

Then, for $0 < h < 1$,

$$
\begin{aligned}
H_h I\ &= \int_h^1 \frac{dx}{x}\ I(\frac{h}{x}) \\
&= \int_h^1 \frac{dx'}{x'}\ I(x'),
\end{aligned}
\tag{6.21}
$$

where $x' = h/x$. Then (4.8) and Lemma 4.1 entail that if I has form F, so does $H_h I$.

Repeated application of this result shows that if $P = x_1 x_2 ... x_p$ then $f(r_0)$ has form F. One first combines $x_{p-1}x_p$ into h_p, then combines $x_{p-2}h_p$ into h_{p-1}, etc.. At each stage the functions I and H have form F, and hence one finally gets (6.16) with $H(h)$ having form F, as required.

The general form of P is not just a single product $r_{i+1}...r_j$: it is a sum of such terms with different values of j, some of which can be multiplied by λ_j or λ_j'. However, these other terms can be brought into the required form by a generalization of the operator technique used above.

Let us again consider first a simple case:

$$
f(r_0) = \int_0^1 dx \int_0^1 dg \int_0^1 dt\ \Theta(r_0 - xt - xg) I(g), \tag{6.22}
$$

where $I(g)$ has form F, and t could be a λ_j. Then

$$f(r_0) = \int_0^1 dx \int_0^2 dh \; \Theta(r_0 - xh)H(h), \tag{6.23}$$

where

$$H(h) = \int_0^1 dg \int_0^1 dt \; \delta(h - t - g)I(g). \tag{6.24}$$

Thus

$$\begin{aligned} f(r_0) \; &= \int_0^1 dx \int_0^1 dh \; \Theta(r_0 - xh)H(h) \\ &+ \int_0^1 dx \int_1^2 dh \; \Theta(r_0 - xh)H(h). \end{aligned} \tag{6.25}$$

For $0 < h < 1$ the function $H(h)$ is

$$\begin{aligned} H(h) \; &= \int_0^1 dg \int_0^1 dt \; \delta(h - t - g)I(g) \\ &= \int_0^1 dg \; I(g)\Theta(h - g)\Theta(1 - (h - g)) \\ &= \int_0^h dg \; I(g), \end{aligned} \tag{6.26}$$

which has form F. Thus the first term in (6.25) gives a contribution $f_1(r_0)$ to $f(r_0)$ that has form F. The second term is, for $0 < r_0 < 1$,

$$\begin{aligned} f_2(r_0) \; &= \int_0^1 dx \int_1^2 dh \; \Theta(r_0 - xh)H(h) \\ &= \int_0^1 dx \int_1^2 dh \; \Theta(\frac{r_0}{x} - h) \int_{h-1}^1 dg \; I(g) \\ &= \int_0^1 dx \int_1^{r_0/x} dh \; \Theta(\frac{r_0}{x} - 1) \; \Theta(2 - \frac{r_0}{x}) \int_{h-1}^1 dg \; I(g) \\ &+ \int_0^1 dx \int_1^2 dh \; \Theta(\frac{r_0}{x} - 2) \int_{h-1}^1 dg \; I(g) \\ &= \int_{r_0/2}^{r_0} dx \int_1^{r_0/x} dh \int_{h-1}^1 dg \; I(g) \\ &+ \int_0^{r_0/2} dx \int_1^2 dh \; \int_{h-1}^1 dg \; I(g) \\ &= r_0 \int_1^2 \frac{dx'}{(x')^2} \int_1^{x'} dh \int_{h-1}^1 dg \; I(g) + (r_0/2) \times \text{const.} \\ &= r_0 \times \text{const.}, \end{aligned} \tag{6.27}$$

which is also of form F.

The two important points are: (1), that the integral operator that reduces a sum $t+g$ to a single h, just like the operator that reduces a product tg to h, preserves form F; and (2), the extra part $h>1$ does not disrupt the argument: it adds only a term $r_0\times$ const.

By taking combinations of these two kinds of operators, and a third kind with t fixed at unity, rather than being integrated over, one can reduce any one of the possible functions $(r_0 - P)$ to $(r_0 - x_1 h)$ combined with an $H(h)$ of form F. Thus all functions $f(r_0)$ of the kind (6.15) will be of the form F , provided we make the simple assignments $1 = a = A = p_s\Omega_j = e_j + 1$. The remaining task is to show that essentially the same result follows even when we do not make these simple assignments. One other problem also needs to be addressed: we have computed the integrals on the variables r_j under the assumption that the variables Ω_j are held fixed, whereas the distortions in the variables Ω_j can depend on the r_j.

By following through the arguments just given, but with the e_j now allowed to be nonnegative integers, one finds that the conclusions are not disrupted: the positive power n of r_0 in $f(r_0)$ can be increased, and the positive power m of $\log r_0$ can be decreased, but changes in the opposite direction do not occur. Hence the singularities are at most weakened.

To deal simultaneously with the problems of the dependence of $A(q, \Omega, r, x)$ upon (r, x), and the dependence of the distortion in Ω upon (r, x), we introduce a sufficiently small number $\epsilon = 1/N > 0$, and divide the domain of integration $0 < x < 1$ in each of the variables x_j and r_j into a sum of N intervals of length ϵ, such that (1), the distortion of the set of Ω variables can be held fixed over each separate product interval, and (2), for any *subset* σ of the set of variables r_j and x_j, and for the corresponding space S formed by the product over σ of the corresponding set of *leading* intervals $0 < (r_j, x_j) < \epsilon$, the dependence of A on these variables can be represented by a power series that converges within S for each point in the space formed by the product over the complementary set of variables of the nonleading intervals $\epsilon < x < 1$. (See Remark 4.3(ii).) The variables can then be re-scaled so that the original integration domains run from 0 to N, and the leading intervals (formerly from 0 to ϵ) now run from from 0

to 1. The earlier arguments can then be applied to the re-scaled problem, with the concept 'form F' replaced by 'form F′': a function of one variable x is said to have form F′ if and only if it is bounded in the interval $0 \leq x \leq 1$, and in $0 < x < 1$ can be written in the form

$$\sum_m A_m(x)(\log x)^m,$$

where the sum is over a finite set of integers m, and each $A_m(x)$ is analytic on $0 < x < 1$. The contributions from the integrations over the nonleading domains $1 < x < N$ do not disrupt the arguments, and formula (4.8) shows that extra factors $n + 1$ are introduced into the denominators at each integration, so that convergence at the level of the integral is, if anything, improved over the original convergence at the level of the integrand. This takes care of these two problems.

The final step is to remove the assumption that the coefficients of the various terms of P are unity: these coefficients are actually the quantities $p_s\Omega_j$.

There is no problem in allowing these coefficients to be strictly-positive Ω-dependent functions: the constants C_{nm}, or the functions A_m, then simply become analytic functions of the variables $p_s\Omega_j$ over these strictly-positive domains. In fact, these coefficients can be continued into the complex domain without affecting the character of the singularity at $r_0 = 0$ provided we keep each coefficient away from the cut along the negative real axis in that variable, and keep the point C in the space of the collection of these coefficients away from all points where $a\lambda + P(C, x_j, r_j) = 0$ for some point in the product of the open domains of integration $0 < \lambda < 1$, and (for all j) $0 < x_j < 1$ and $0 < r_j < 1$. Here $a = p_s\Omega_i$.

The points in the domain of integration over the variables r_j, x_j, Ω_j that contribute to the singularity at $\varphi = 0$ are points where each of the three star-line factors is evaluated at, or very close to, the associated pole. The arguments in ref. 2 show that in this region the first of the variables $p_s\Omega_j$, namely $p_s\Omega_i = a$ can be shifted into upper-half plane $\mathrm{Im}a > 0$, and the collection of contours C can be distorted so that $a\lambda + P$ is shifted into the upper-half-plane provided $0 < \lambda < 1$ and, for all j, $0 < r_j < 1$ and $0 < x_j < 1$. This is exactly the condition that is needed to justify the extension of the results obtained above for positive real coefficients to the complex points of interest.

The dependence of the distortion of the contour on λ needs to be described. When one introduces the nonmeromorphic couplings, and hence the $\int d\lambda$, into the formula, the Landau matrix acquires a new column, the $d\lambda$ column. However, the parameter λ enters in an almost trivial way: the pole residues associated with the side s of the triangle into which the vertex associated with λ is coupled are changed from $p_s(\Omega_j + ...)$ into $(p_s + \lambda_i k_i)(\Omega_j + ...)$, and the pole denominator $(p_s)^2 - m^2$ is changed to $(p_s + \lambda_i k_i)^2 - m^2$. The new set of Landau equations can be satisfied at each of the two end points $\lambda_i = 1$ and $\lambda_i = 0$. These two solutions correspond to diagrams in which the vertex associated with λ_i is placed at one end or the other of the side s of the triangle. Both solutions to the triangle-diagram equations exist, and, because of the null contributions in all $d\Omega_j$ columns, the two solutions yield two different ways of distorting the Ω_j contours, Δ_1 and Δ_0, the first corresponding to $\lambda_i = 1$, the second corresponding to $\lambda_i = 0$. An allowed distortion that gives these two cases and smoothly interpolates to the intermediate values of λ_i is $\lambda_i\Delta_1 + (1 - \lambda_i)\Delta_0$. It keeps the imaginary part of the pole denominator strictly positive (near the zero of the real part) for all values of λ_i in the domain $0 \leq \lambda_i \leq 1$. This distortion, or some approximation to it, can be used in the argument given above.

For the remaining integrations on the dr_j ($j \leq i$) one uses Propositions 4.4(ii) and 4.5, and Remark 4.3(ii). This gives for the singular part of $F(q)$ at $\widehat{q}$ a function of form F$'$, in some appropriately scaled variable $\varphi(q)$, multiplied by an analytic function of q.

7. Computation for Self-Energy Case.

Contributions from photon lines j coupled nonmeromorphically on both ends into the same side s of the triangle were excluded from the discussion in section 6. For these values of j one can, in order to exclude double counting, impose the condition $\lambda_j \geq \lambda'_j$, where λ_j is the λ-parameter associated with the nonmeromorphic coupling on the tail of photon line j, and λ'_j is the λ-parameter associated with the nonmeromorphic coupling on the head of line j. Momentum k_j flows along line j from its tail to its head, according to our conventions.

The formulas of section 2 of Ref. 1 refer to momentum k_j flowing out of the charged-particle line at the tail of the photon line j. The coupling at the head can be treated like the coupling at the tail, but with a reversal of the sign of k_j. Then the effect of the two couplings into the same side s is to replace p_s by $p_s + (\lambda_j - \lambda'_j)k_j$, and to integrate on λ_j from zero to one and on λ'_j from zero to λ_j. The condition Im $p_s\Omega_j > 0$ then retains its usual from. The reduction of the domain of integration does not upset the arguments of section 6.

To bring this case into accord with section 6 we use the following transformations:

$$
\begin{aligned}
&\int_0^1 d\lambda \int_0^\lambda d\lambda' f((\lambda - \lambda')k) \\
= &\int_0^1 \lambda d\lambda \int_0^1 d\lambda'' f(\lambda(1-\lambda'')k) \\
= &\int_0^1 \lambda d\lambda \int_0^1 d\lambda''' f(\lambda\lambda''' k) \\
= &\int_0^1 dh f(hk) \int_0^1 \lambda d\lambda \int_0^1 d\lambda''' \delta(h - \lambda\lambda''') \\
= &\int_0^1 dh f(hk) \int_0^1 d\lambda \int_0^1 d\lambda''' \delta(\frac{h}{\lambda} - \lambda''') \\
= &\int_0^1 dh f(hk) \int_h^1 d\lambda \\
= &\int_0^1 dh f(hk)(1-h).
\end{aligned}
$$

The variable h plays the role played by λ in section 6.

References

1. T. Kawai and H.P. Stapp, *Quantum Electrodynamics at Large Distances I: Extracting the Correspondence–Principle Part.* Lawrence Berkeley Laboratory Report LBL 35971. Submitted to Phys. Rev.

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3. H.P. Stapp, *in* Structural Analysis of Collision Amplitudes, ed. R. Balian and D. Iagolnitzer, North Holland, New York (1976)